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A QUASILINEAR PARABOLIC EQUATION DESCRIBING THE ELONGATION OF T--ETC(U)

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A QUASILINEAR PARABOLIC EQUATION  
DESCRIBING THE ELONGATION OF THIN  
FILAMENTS OF POLYMERIC LIQUIDS.

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A QUASILINEAR PARABOLIC EQUATION DESCRIBING THE  
ELONGATION OF THIN FILAMENTS OF POLYMERIC LIQUIDS

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ABSTRACT

We study the equation

$$\rho \ddot{u} = 3\eta \frac{\partial^2}{\partial x \partial t} \left( -\frac{1}{u_x} \right) + \frac{\partial}{\partial x} \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds$$

where  $u(x,t)$  is a real valued function of  $x \in [-1,1]$  and  $t \in \mathbb{R}$ , with the boundary condition

$$3\eta \frac{\partial}{\partial t} \left( -\frac{1}{u_x} \right) + \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds = f(t)$$

at  $x = \pm 1$ . This equation is derived as a model for the elongation of thin filaments of polymeric liquids,  $u$  denoting the position of a fluid particle in space,  $a$  the memory kernel, and  $f$  the force acting on the ends of the filament. We study the evolution of  $u$ , assuming the initial condition  $u(x, t = -\infty) = x$ . It is shown that under appropriate conditions on  $a$  and  $f$  the boundary condition can be uniquely resolved with respect to  $u_x$ . The full problem is transformed in such a way that it is approachable by the Sobolevskii theory of quasilinear parabolic equations. This yields the existence of solutions to the initial value problem on sufficiently small time intervals. Moreover, we show that if  $f(t)$  converges to zero exponentially as  $t \rightarrow +\infty$  and is small in an appropriate norm, there exists a solution globally in time, which approaches a stationary limit as  $t \rightarrow +\infty$ .

AMS (MOS) Subject Classifications: 34G20, 35K55, 35Q20, 45K05, 47H20, 76A10

Key Words: Viscoelastic liquids, quasilinear parabolic equations, nonlinear semigroups

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## SIGNIFICANCE AND EXPLANATION



The equation under study - stated in the abstract and derived from physical principles in this paper - describes the elongation of a filament of a polymeric liquid subjected to a force  $f$  at both ends. The liquid is assumed to satisfy certain accepted "rubberlike liquid" constitutive relations, and the filament is assumed to be thin, which permits a reduction of the problem to one space dimension. The unknown variable  $u$  denotes the position of a fluid particle at time  $t$ , which was at position  $x$  at  $t = -\infty$ , i.e., before the deformation started, we have  $u(x, -\infty) = x$ . In this paper the equation under study is transformed in such a way that it fits into the framework of the general mathematical theory for "quasilinear parabolic equations". This makes it possible to prove that for any given "initial condition" a solution exists at least on a certain time interval. (It is a part of the analysis to discover what is an appropriate meaning of "initial condition" to be associated with the problem under study). Moreover, we shall prove that for forces  $f(t)$ , which approach zero exponentially for  $t \rightarrow +\infty$  and are small in a suitable sense, there is a solution for all times  $t$ ,  $-\infty < t < +\infty$ , and this solution approaches a stationary limit as  $t \rightarrow +\infty$ .

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A QUASILINEAR PARABOLIC EQUATION DESCRIBING THE  
ELONGATION OF THIN FILAMENTS OF POLYMERIC LIQUIDS

M. Renardy

0. INTRODUCTION.

We study the following problem occurring in polymer processing: A thin filament of a viscoelastic liquid is subjected to a force  $f$  acting on its ends as shown in the diagram:



We investigate the temporal evolution of the displacement. The equations that our analysis is based on involve the "rubberlike liquid" constitutive assumption for the stress-strain law [3] and certain approximations based on the thinness of the filament, which allow the reduction to a spatially one-dimensional problem. Using these assumptions, we shall derive the following equation

$$\rho \ddot{u} = 3\eta \frac{\partial^2}{\partial x \partial t} \left( -\frac{1}{u_x} \right) + \frac{\partial}{\partial x} \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds \quad (0.1)$$

where  $u(x,t)$  is a real valued function of  $x \in [-1,1]$  and  $t \in \mathbb{R}$ . As usual, a subscript  $x$  denotes partial differentiation w.r. to  $x$  and "dot" denotes partial differentiation w.r. to  $t$ . The arguments  $(x,t)$  are suppressed unless needed for proper understanding. (0.1) is supplemented by the nonlinear Neumann boundary condition

$$3\eta \frac{\partial}{\partial t} \left( -\frac{1}{u_x} \right) + \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds = f(t) \quad (0.2)$$

at  $x = \pm 1$ .

In these equations  $u(x,t)$  denotes the position at time  $t$  of a fluid particle, which is at the position  $x$  in a certain reference state. This reference state will be identified with the state of the fluid at  $t = -\infty$ , i.e. we have  $u(x, t = -\infty) = x$ .  $\rho$  denotes the density of the fluid,  $\eta$  the viscosity, and  $f$  the force acting on the

ends of the filament. The memory kernel  $a : [0, \infty) \rightarrow \mathbb{R}$  will be assumed to have the following properties, which we shall refer to as assumptions (a):

(i)  $a$  has the representation

$$a(t) = \int e^{-\lambda t} d\mu(\lambda) \quad (0.3)$$

where  $\mu$  is a complex valued Borel measure on the complex plane  $\mathbb{C}$  such that  $1 \in L^1(\mu)$  (i.e.  $1$  is integrable w.r. to the total variation of  $\mu$ ), and  $\text{supp } \mu$  is contained in  $\{\lambda \in \mathbb{C} \mid -\varphi \leq \arg \lambda \leq \varphi, |\lambda| \geq \varepsilon\}$  for some  $\varphi < \frac{\pi}{2}$  and  $\varepsilon > 0$ . Since  $a$  is real, we may and will assume that  $d\mu(\bar{\lambda}) = \overline{d\mu(\lambda)}$ .

(ii)  $a(t) \geq 0$  for  $t \in [0, \infty)$ .

(iii)  $a$  is monotonely decreasing.

Note that (i) implies in particular that  $a$  is continuous and that  $|a|$  can be estimated by a decaying exponential. The motivation for assumptions (a) will become apparent later in the paper. The sectorial condition for  $\text{supp } \mu$  is needed to make the problem fit into the theory of parabolic equations, (ii) and (iii) will have important implications for the spectra of certain linear operators. In physical theories derived from "molecular network" or "bead-spring" models (see [4] and the references in [3], ch. 6)  $a$  turns out to be a finite sum of decaying exponentials. This is clearly a special case of assumptions (a),  $\mu$  in this case being a finite sum of Dirac measures located on the real axis.

The boundary condition (0.2) agrees precisely with the equation describing the evolution of the length of the filament when inertial forces are neglected. This problem has been discussed previously by Lodge, McLeod and Nohel in [5] and by the author in [7]. Lodge, McLeod and Nohel consider the solution as known for  $t < 0$  and assume it is nondecreasing. They then assume  $f = 0$  for  $t > 0$  and study existence, asymptotic behaviour and various monotonicity properties of solutions. In [7] the force  $f$  is a given continuous function  $\mathbb{R} \rightarrow \mathbb{R}$ . It is assumed that  $f$  converges to zero exponentially as  $t \rightarrow -\infty$ , and that either  $f$  converges to 0 exponentially as  $t \rightarrow +\infty$  and is small in a suitable norm, or the size of  $f$  is arbitrary, but  $f(t)$

vanishes identically for  $t$  greater than some finite  $t_0$ . (In the latter case we need the additional assumption that  $\text{supp } \mu$  is on the real axis; in fact in [7] we assumed that  $\mu$  was a finite sum of Dirac measures on the real axis, but the same ideas can be applied to the more general situation as we demonstrate below). In both cases it is proved that, given the initial condition  $u_x(t = -\infty) = 1$ , a unique positive solution exists for all times  $t$ , and moreover  $\lim_{t \rightarrow \infty} u_x(t)$  exists and is strictly positive. Whereas the arguments in [5] rely mainly on monotonicity properties, the main tools in [7] are the implicit function theorem and the use of Liapunov functions.

The present paper will be arranged as follows: In §1 we explain the basic physical laws and the approximations leading to (0.1), (0.2). We start from the basic laws of continuum mechanics, using the "rubberlike liquid" constitutive relation. The equation of motion in the interior of the filament and the boundary conditions on the lateral surface are then solved formally by a power series expansion with respect to a "thinness parameter" in an analogous manner as was done in the theory of thin elastic rods [6]. The first order terms in this expansion lead to (0.1). The formal expansion does not in general fit given boundary conditions at the ends of the filament, and one is confronted with a "boundary layer" problem. Since we are only interested in a first order approximation, we shall not deal with this situation here. Instead, we consider the balance of force, taking into account only terms not involving the small parameter. This leads to (0.2). §2 summarizes the results of [7] concerning (0.2) as explained above, taking into account the modifications required by the more general assumptions on  $a$ . As a result, we may subsequently consider  $u_x$  as being given on the boundary. In §3 and §4 we finally deal with the full problem (0.1), (0.2). Using (0.3), this problem is transformed in such a way that it fits into the abstract theory of quasi-linear parabolic equations introduced by Sobolevskii [2], [8]. An "initial condition" in the evolution problem so defined will not necessarily involve the whole history of  $u$ , but only certain of its moments, the choice of which depends on the support of  $\mu$ . In §5 we shall explain this transformation and as a consequence of the Sobolevskii

theory obtain the existence and uniqueness of solutions to the initial value problem locally in time. §4 deals with the case where  $f$  converges to zero exponentially as  $t \rightarrow +\infty$  and is small. We assume the filament is undeformed ( $u = x$ ) at  $t = -\infty$ . It will be shown that a solution of the full problem exists globally in time, which approaches a stationary limit as  $t \rightarrow +\infty$ .



# 1. DERIVATION OF THE BASIC EQUATIONS.

We assign to each point in the fluid two different sets of coordinates: By  $(\zeta^1, \zeta^2, \zeta^3)$  we denote "body coordinates", i.e. coordinates labelling a specific particle in the fluid. These coordinates can be identified with the position of the particle in space, when the fluid is in a certain "reference state". (It will later be convenient to take as a reference state the state of the fluid at time  $t = -\infty$ ). On the other hand  $(y^1, y^2, y^3)$  will denote coordinates labelling a point in space. We are interested in finding trajectories of fluid particles, i.e. a functional dependence  $y^i(\zeta^1, \zeta^2, \zeta^3, t)$ . In our exposition of the equations describing this functional dependence we follow Lodge [3]. (For a summary of the relevant equations, see p. 206-207).

To each point  $(\zeta^1, \zeta^2, \zeta^3)$  there is assigned a body metric tensor  $\gamma$  and a body stress tensor  $\pi$ .  $\gamma$  is defined by the relation

$$\gamma_{ij} = \frac{\partial y^r}{\partial \zeta^i} \cdot \frac{\partial y^r}{\partial \zeta^j}$$

$\pi$  is related to  $\gamma$  by a constitutive law, which expresses the specific properties of the material. We use the "rubberlike liquid" constitutive relation ([3], p. 143):

$$\pi^{ij} + p\gamma^{ij} = -\eta \frac{\partial \gamma^{ij}}{\partial t} + \int_{-\infty}^t a(t-s) \gamma^{ij}(s) ds$$

where  $\gamma^{ij}$  denote the components of  $\gamma^{-1}$ , i.e.  $\gamma^{ij} \gamma_{jk} = \delta_k^i$ ;  $\eta$  is a positive material constant called the viscosity, and  $a$  is a given function, which will always be assumed to satisfy assumptions (a) stated in the introduction.  $p$  is an unknown variable having the physical significance of a pressure. The introduction of this variable is necessary, since we assume the fluid is incompressible

$$\det \gamma = 1 \quad (1.1)$$

The evolution of  $y^k$  is determined by Newton's law, which for a Cartesian space coordinate system takes the form

$$\rho \ddot{y}^k = \frac{\partial y^k}{\partial \zeta^i} \left\{ \frac{\partial}{\partial \zeta^s} \pi^{si} + \Gamma_{rs}^i \pi^{rs} \right\} \quad (1.2)$$

The  $\Gamma_{rs}^i$  denote the Riemann-Christoffel symbols associated with the metric tensor  $\gamma$ :

$$\Gamma_{rs}^i = \frac{1}{2} \gamma^{ij} \left\{ \frac{\partial \gamma_{jr}}{\partial \zeta^s} + \frac{\partial \gamma_{js}}{\partial \zeta^r} - \frac{\partial \gamma_{rs}}{\partial \zeta^j} \right\}$$

Equations (1.1) and (1.2) have to be supplemented by boundary conditions referring to either the displacement or the stresses on the boundary of the liquid. We shall here deal with stress conditions. Let  $Y^k$  denote the components of surface traction referred to space coordinates. Then the boundary conditions on a surface  $\zeta^l = \text{const.}$  are given by

$$\pi^{il} \frac{\partial y^k}{\partial \zeta^i} (\gamma^{ll})^{-1/2} = Y^k \quad (1.3)$$

In the problem of the elongated filament the surface traction on the lateral surface is zero, whereas at the ends there is a longitudinal surface traction equal to  $f$  divided by the cross-sectional area of the filament. For convenience, we let  $\zeta^1$  and  $y^1$  resp. denote the coordinate in the direction of the filament and  $\zeta^2, \zeta^3$  and  $y^2, y^3$  resp. the transversal coordinates. It is assumed that in the undeformed reference state (i.e. at  $t = -\infty$ ) the filament is cylindrical and axi-symmetric, i.e. in this state we have  $y^i = \zeta^i$ , where  $\zeta^1$  (by appropriate normalization of length scale) ranges from  $-1$  to  $1$ , and  $r = \sqrt{(\zeta^2)^2 + (\zeta^3)^2}$  ranges from  $0$  to  $\delta$ , the radius of the filament. Then, for small  $\delta$ , equations (1.1) and (1.2) and the boundary conditions on the lateral surface can formally be solved by a series expansion in powers of  $r$  and  $\delta$ . This expansion is analogous to that used by Nariboli [6] for the problem of longitudinal elastic waves in a thin rod.

We put  $\hat{\zeta}^2 = \frac{\zeta^2}{\delta}$ ,  $\hat{\zeta}^3 = \frac{\zeta^3}{\delta}$ ,  $\hat{r} = \frac{r}{\delta}$ , so that the lateral surface now corresponds to  $\hat{r} = 1$ . We make the following ansatz

$$y^1(\zeta^1, \delta \hat{\zeta}^2, \delta \hat{\zeta}^3) = \sum_{v=0}^{\infty} \delta^{2v} P_v(\zeta^1, \hat{r}^2)$$

$$y^2(\zeta^1, \delta \hat{\zeta}^2, \delta \hat{\zeta}^3) = \delta \hat{\zeta}^2 \sum_{v=0}^{\infty} \delta^{2v} Q_v(\zeta^1, \hat{r}^2)$$

$$y^3(\zeta^1, \delta \hat{\zeta}^2, \delta \hat{\zeta}^3) = \delta \hat{\zeta}^3 \sum_{v=0}^{\infty} \delta^{2v} Q_v(\zeta^1, \hat{r}^2)$$

$$p(\zeta^1, \delta \hat{\zeta}^2, \delta \hat{\zeta}^3) = \sum_{v=0}^{\infty} \delta^{2v} R_v(\zeta^1, \hat{r}^2)$$

where  $P_v, Q_v, R_v$  are polynomials of  $v$ th degree in  $\hat{r}^2$ . This ansatz is inserted into (1.1), (1.2) and the lateral boundary conditions, which are supposed to be satisfied for all values of  $\delta$ . Formally this yields an infinite set of equations for the coefficients of  $P_v, Q_v$  and  $R_v$ . We are only interested in deriving an equation for the first term  $P_0(\zeta^1)$ , and we shall in the following only carry out the series expansion as far as needed for this purpose.

When terms up to  $O(\delta)$  are taken into account, we find for the metric tensor

$$\gamma = \begin{pmatrix} \left(\frac{\partial P_0}{\partial \zeta^1}\right)^2 & \delta \hat{\zeta}^2 \cdot \varphi & \delta \hat{\zeta}^3 \cdot \varphi \\ \delta \hat{\zeta}^2 \cdot \varphi & Q_0^2 & 0 \\ \delta \hat{\zeta}^3 \cdot \varphi & 0 & Q_0^2 \end{pmatrix}$$

$$\text{where } \varphi = \frac{\partial P_0}{\partial \zeta^1} \cdot \frac{\partial P_2}{\partial \hat{r}} \hat{r}^{-1} + \frac{\partial Q_0}{\partial \zeta^1} Q_0$$

Putting  $\delta = 0$ , we find from (1.1)

$$\left(\frac{\partial P_0}{\partial \zeta^1}\right)^2 Q_0^4 = 1 \quad (1.4)$$

Next we consider the boundary conditions on the lateral surface. At a boundary point where  $\hat{\zeta}^3 = 0$  these yield the equations  $\pi^{21} = \pi^{22} = \pi^{23} = 0$ . (Because of the radial symmetry it suffices to consider these boundary points; if the traction vanishes there,

it does so everywhere else.)  $\pi^{23}$  vanishes identically as a result of the radial symmetry. For  $\pi^{22}$  we obtain the following terms of the order 0(1)

$$\pi^{22} = -R_0 Q_0^{-2} - \eta \frac{\partial}{\partial t} (Q_0^{-2}) + \int_{-\infty}^t a(t-s) Q_0^{-2}(s) ds = 0 \quad (1.5)$$

Finally we have

$$\pi^{21} = -p\gamma^{21} - \eta \frac{\partial}{\partial t} (\gamma^{21}) + \int_{-\infty}^t a(t-s) \gamma^{21}(s) ds = 0 \quad (1.6)$$

All solutions we are going to consider shall satisfy  $\lim_{t \rightarrow \infty} p(t) = \int_{-\infty}^0 a(-s) ds$  and  $\lim_{t \rightarrow \infty} \gamma^{21}(t) = 0$ , the convergence being exponential. If (a) holds, it is then not difficult to prove that the only solution to (1.6) satisfying the specified conditions is  $\gamma^{21} = 0$ . In the first order in  $\delta$  this yields  $\varphi = 0$ . The law of motion (1.2) now yields the following equation for  $P_0 \left( \gamma^{11} = \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2} \right)$ :

$$\begin{aligned} \rho \ddot{P}_0 &= \frac{\partial P_0}{\partial \zeta^1} \left\{ \frac{\partial}{\partial \zeta^1} \pi^{11} + \Gamma_{11}^1 \pi^{11} \right\} \\ &= \frac{\partial P_0}{\partial \zeta^1} \left\{ \frac{\partial}{\partial \zeta^1} \left( -R_0 \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2} - \eta \frac{\partial}{\partial t} \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2} + \int_{-\infty}^t a(t-s) \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2}(s) ds \right) \right. \\ &\quad \left. + \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-1} \frac{\partial^2 P_0}{(\partial \zeta^1)^2} \left( -R_0 \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2} - \eta \frac{\partial}{\partial t} \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2} + \int_{-\infty}^t a(t-s) \left( \frac{\partial P_0}{\partial \zeta^1} \right)^{-2}(s) ds \right) \right\} \end{aligned}$$

In order to simplify notation, we shall henceforth write  $u$  for  $P_0$  and  $x$  for  $\zeta^1$ . The last equation now yields (0.1) after a few manipulations, when (1.5) and (1.4) are used to express  $R_0$  in terms of  $u$ .

Finally we have to specify boundary conditions at the ends of the filament. As noted in [6], the asymptotic expansion which we used for the interior problem generally fails near the ends, and a "boundary layer" has to be taken into account. The boundary layer is discussed in a forthcoming paper by Reiss, which is referenced in [1], but not available yet. We are here only concerned with a first order approximation, and we shall ignore boundary layer effects. Instead, we take care of the force balance in the zeroth order with respect to  $\delta$ . Namely, if one formally inserts our expansion

into the boundary conditions at the ends, it is seen that all traction components transversal to the direction of the filament are  $O(\delta)$ . The longitudinal traction component gives the following terms of order  $O(1)$ :

$$\begin{aligned}\tau &= \pi l l \frac{\partial p_0}{\partial \zeta} (\gamma l l)^{-1/2} = \pi l l \left( \frac{\partial p_0}{\partial \zeta} \right)^2 \\ &= 3\eta \frac{\dot{u}_x}{u_x} + \int_{-\infty}^t a(t-s) \left( \frac{u_x^2(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x(t)} \right) ds\end{aligned}$$

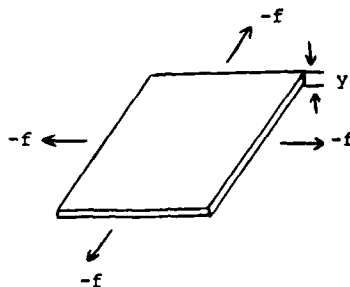
Since the cross-sectional area of the filament is in first approximation equal to  $u_x^{-1}$ , we shall require that  $\tau = f \cdot u_x$ . This yields (0.2).

## 2. THE BOUNDARY PROBLEM.

In this section we consider the problem of solving (0.2) for  $u_x$ , when  $f$  is given. The results we present slightly generalize those of [7], allowing for the more general class of kernels  $a$  satisfying assumptions (a). In order to simplify notation, we write  $y$  for  $u_x$ . Instead of (0.2) we study the slightly more general problem

$$3\eta\dot{y} + \int_{-\infty}^t a(t-s) \left( \frac{y^3(t)}{y^2(s)} - y(s) \right) ds = f(t)y^\alpha \quad (2.1)$$

where  $0 < \alpha < 3$ . (0.2) corresponds to  $\alpha = 2$ , and, as explained in [7], the case  $\alpha = \frac{1}{2}$  is also physically interesting, namely, it describes the deformation of a sheet of the polymer, when inertia are neglected:



We put  $g(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} y^{-2}(s) ds$ ,  $h(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} y(s) ds$ ,

$$\gamma(\lambda) = g(\lambda)y^2, \quad \delta(\lambda) = h(\lambda)y^{-1}$$

Then (2.1) is equivalent to either of the following systems

$$\begin{aligned} 3\eta\dot{\gamma} &= \int (\delta(\lambda) - \gamma(\lambda)y^3) d\mu(\lambda) + fy^\alpha \\ \dot{g}(\lambda) &= -\lambda g(\lambda) + y^{-2} \\ \dot{h}(\lambda) &= -\lambda h(\lambda) + y \end{aligned} \quad (2.2)$$

$$\begin{aligned} 3\eta\dot{\gamma} &= y \cdot \int (\delta(\lambda) - \gamma(\lambda)) d\mu(\lambda) + fy^\alpha \\ \dot{\gamma}(\lambda) &= -\lambda\gamma(\lambda) + 1 - \frac{2}{3\eta} \gamma(\lambda) \int (\gamma(\lambda) - \delta(\lambda)) d\mu(\lambda) + \frac{2}{3\eta} \gamma(\lambda) fy^{\alpha-1} \\ \dot{\delta}(\lambda) &= -\lambda\delta(\lambda) + 1 + \frac{1}{3\eta} \delta(\lambda) \int (\gamma(\lambda) - \delta(\lambda)) d\mu(\lambda) - \frac{1}{3\eta} \delta(\lambda) fy^{\alpha-1} \end{aligned} \quad (2.3)$$

Both forms will be used in the following. Equations (2.2) or (2.3) will be regarded as evolution problems in the space  $X = \mathbb{R} \times (L^s(\mu))^2$  ( $1 \leq s < \infty$ ). Here  $L^s(\mu)$  denotes the space of all (equivalence classes of) functions  $g : \mathbb{C} \rightarrow \mathbb{C}$  such that  $g(\bar{z}) = \overline{g(z)}$  and  $|g|^s$  is integrable w.r. to the total variation of  $\mu$ . Clearly, the right side of (2.2) or (2.3) is the sum of an analytic generator and a smooth nonlinear term.

A trivial solution for  $f = 0$  is given by  $y = 1$ ,  $g(\lambda) = h(\lambda) = \gamma(\lambda) = \delta(\lambda) = \frac{1}{\lambda}$ , and we are interested in solutions converging to this trivial solution as  $t \rightarrow -\infty$ . As a first step we investigate the spectral properties of the linearization of (2.2) (of course (2.3) gives the same result) at this point, i.e. we study the inhomogeneous linear equation

$$\begin{aligned} 3\eta\beta y - \int (h(\lambda) - g(\lambda)) d\mu(\lambda) + 3y \int \frac{1}{\lambda} d\mu(\lambda) &= 3\eta\varphi \\ \beta g(\lambda) + \lambda g(\lambda) + 2y &= \psi_1(\lambda) \\ \beta h(\lambda) + \lambda h(\lambda) - y &= \psi_2(\lambda) \end{aligned} \quad (2.4)$$

If  $-\beta$  is not in the support of  $\mu$ , the last two equations can be resolved with respect to  $g(\lambda)$  and  $h(\lambda)$ . This inserted into the first equation of (2.4) yields

$$\begin{aligned} 3\eta\beta y - 3y \int \frac{1}{\lambda + \beta} d\mu(\lambda) + 3y \int \frac{1}{\lambda} d\mu(\lambda) \\ = 3\eta\varphi + \int \frac{\psi_2(\lambda) - \psi_1(\lambda)}{\lambda + \beta} d\mu(\lambda) \end{aligned}$$

Hence the resolvent exists at  $\beta$ , iff  $-\beta$  is not in the support of  $\mu$  and

$$\rho(\beta) := 3\eta\beta - 3 \int \left( \frac{1}{\lambda + \beta} - \frac{1}{\lambda} \right) d\mu(\lambda) \neq 0$$

Clearly,  $\rho(\beta)$  vanishes for  $\beta = 0$ . Namely, we have

$$\rho(0) = 3\eta \left( \eta + \int \frac{1}{\lambda(\lambda + \beta)} d\mu(\lambda) \right)$$

Using the relationship between  $\mu$  and the kernel  $a$ , we find

$$\int \frac{1}{\lambda(\lambda + \beta)} d\mu(\lambda) = \int_0^\infty a(t) \frac{1 - e^{-\beta t}}{\beta} dt$$

For  $\beta \neq 0$ , the real part of this expression is given by

$$\int_0^{\infty} \frac{a(t)}{|\beta|^2} \{ \operatorname{Re} \beta (1 - e^{-t \operatorname{Re} \beta} \cos(t \operatorname{Im} \beta)) + \operatorname{Im} \beta e^{-t \operatorname{Re} \beta} \sin(t \operatorname{Im} \beta) \} dt$$

If  $\operatorname{Re} \beta \geq 0$ , condition (a) (ii) implies that the first contribution is positive, and condition (a) (iii) implies that the second contribution is positive, too. Hence

$\operatorname{Re} \int \frac{1}{\lambda(\lambda + \beta)} d\mu(\lambda) \geq 0$ , whence certainly  $\rho(\beta) \neq 0$ .

For easier reference, let us put  $Y = (y, g, h) \in \mathbb{R} \times (L^2(u))^2$  in (2.2) and write (2.2) in the form

$$\dot{Y} = L(Y - Y_0) + N(Y - Y_0, f) \quad (2.5)$$

where  $L$  denotes the linearization of the right side at the trivial solution

$Y_0 = (1, \frac{1}{\lambda}, \frac{1}{\lambda})$ . Analogously, we put  $Y' = (y, \gamma, \delta)$  and write (2.3) in the form

$$\dot{Y}' = L'(Y' - Y_0) + N'(Y' - Y_0, f) \quad (2.6)$$

We have just proved

Proposition 2.1:

The spectrum of  $L$  (or  $L'$ ) consists of the algebraically simple eigenvalue 0 (geometric simplicity is immediate, and algebraic simplicity follows from the fact that the resolvent has a first order pole) and a remainder contained in the left half plane. Moreover, the restriction of  $L$  to the range of  $L$  generates an analytic semigroup of negative type.

Before we can state our theorems, we must first define some spaces of functions.

Definition 2.2:

Let  $Z$  be a Banach space and  $\sigma$  a positive real number. Then

$$X_n^\sigma(Z) := \{v \in C^n(\mathbb{R}, Z) \mid \lim_{t \rightarrow \infty} e^{\sigma|t|} \|v^{(k)}(t)\| = 0 \text{ for } k = 0, 1, \dots, n;\$$

$v^{(k)}$  denoting the  $k$ th derivative\}



$$Y_n^\sigma(Z) = \{v \in C^n(\mathbb{R}, Z) \mid \lim_{t \rightarrow \infty} e^{\sigma t} \|v^{(k)}(t)\| = 0 \text{ for } k = 1, 2, \dots, n\},$$

$$\lim_{t \rightarrow \infty} e^{-\sigma t} \|v(t)\| = 0, \quad \lim_{t \rightarrow \infty} v(t) =: v(\infty) \text{ exists and}$$

$$\lim_{t \rightarrow \infty} e^{\sigma t} \|v(t) - v(\infty)\| = 0.$$

A natural norm in  $X_n^\sigma$  is

$$\|v\| = \sum_{k=0}^n \sup_{t \in \mathbb{R}} e^{\sigma |t|} \|v^{(k)}(t)\|$$

A natural norm in  $X_n^\sigma$  is

$$\|v\| = \sum_{k=1}^n \sup_{t \in \mathbb{R}} e^{\sigma |t|} \|v^{(k)}(t)\| + \sup_{t \leq 0} e^{-\sigma t} \|v(t)\| + \|v(\infty)\| + \sup_{t \geq 0} e^{\sigma t} \|v(t) - v(\infty)\|$$

**Theorem 2.3:**

Let  $\sigma > 0$  be small enough. Then the following holds: If  $f \in X_n^\sigma(\mathbb{R})$  has sufficiently small norm, (2.6) has a unique solution  $Y'$  satisfying

$$\tilde{Y} = Y' - Y_0 \in Y_n^\sigma(\mathbb{R}) \times (X_n^\sigma(L^S(\mu)))^2. \quad Y' \text{ depends smoothly on } f.$$

**Proof:**

We rewrite (2.6) in the form

$$G(\tilde{Y}, f) = \tilde{Y} - \left(\frac{d}{dt} - L'\right)^{-1} N'(\tilde{Y}, f) = 0 \quad (2.7)$$

It is a consequence of Proposition 2.1 that  $\left(\frac{d}{dt} - L'\right)^{-1}$  maps  $X_n^\sigma(\mathbb{R}) \times X_n^\sigma(L^S(\mu))^2$  into  $Z_n := Y_n^\sigma(\mathbb{R}) \times (X_n^\sigma(L^S(\mu)))^2$ . Hence  $G$  is a smooth mapping from  $Z_n \times X_n^\sigma(\mathbb{R})$  into  $Z_n$  and we have  $D_{\tilde{Y}} G(0, 0) = \text{id}$ . By the implicit function theorem, (2.7) can therefore be resolved with respect to  $\tilde{Y}$  in a sufficiently small neighbourhood of  $(0, 0)$ .

$\tilde{Y}$  is clearly unique within that neighbourhood. We want to show that it is in fact unique within the class of all functions converging to zero as  $T \rightarrow -\infty$ . To see this, let us first consider functions  $\tilde{Y}$  satisfying  $\lim_{t \rightarrow -\infty} e^{-\sigma t} \|\tilde{Y}\| = 0$  for some  $\sigma$  between 0 and  $\sigma_0$ . If  $Y$  is such a function, then certainly  $e^{-\sigma t} \|Y(t)\|$  is smaller than  $\epsilon$  on some interval  $(-\infty, t_1]$ . We can now apply an analogous implicit

function argument as above, but rather than considering functions on all of  $\mathbb{R}$ , we consider only function on  $(-\infty, t_1]$ . From this we see that  $\bar{y}$  is unique in the class of all functions that approach zero exponentially as  $t \rightarrow -\infty$ . Finally, if we assume  $\bar{y}$  converges to zero at all, it can be seen from the last two equations of (2.3) that  $\gamma - \frac{1}{\lambda}$  and  $\delta - \frac{1}{\lambda}$  converge to zero exponentially, because if only these two equations are considered, the zero eigenvalue in the linearization does not occur. From the first equation of (2.3) we find that  $\dot{y}$  converges to zero exponentially, and hence the convergence of  $y$  to its limit has to be exponential, too.

If further restrictions are made on  $\mu$ , a global result can be proved that does not rely on the smallness of  $f$ .

Theorem 2.4:

In addition to (a), assume  $\text{supp } \mu$  is contained in the real axis and  $\mu$  is positive real. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and such that  $\lim_{t \rightarrow \infty} e^{-\alpha t} f(t) = 0$  for some  $\alpha > 0$  and  $f(t) = 0$  for  $t \geq t_0$ . For any such  $f$ , equation (2.3) has a unique solution satisfying  $\lim_{t \rightarrow \infty} Y'(t) = Y_0$ . This solution exists globally in time, moreover,

$\lim_{t \rightarrow \infty} Y'(t) = (y(\infty), \frac{1}{\lambda}, \frac{1}{\lambda})$  exists and  $y(\infty) > 0$ .

Proof:

From the arguments in the proof of the last theorem we already know the existence and uniqueness of a solution on some interval  $(-\infty, t_1]$ . In order to prove that the solution exists globally in time, it is more convenient to look at (2.2) rather than at the equivalent equation (2.3). Solutions of (2.2) continue to exist as long as  $y$  stays away from zero or infinity. From the second and third equation of (2.2) one obtains positive lower bounds for  $\int q(\lambda) d\mu(\lambda)$  and  $\int h(\lambda) d\mu(\lambda)$  in every finite time interval, provided that  $y$  remains positive, and these bounds do not depend on any estimate for  $y$ . Hence, if  $y$  becomes too large,  $y^3 \cdot \int q(\lambda) d\mu(\lambda)$  will dominate over  $fy^2$  and also over  $\int h(\lambda) d\mu(\lambda)$  (the latter being less than some constant times  $\max_{t \in (-\infty, t]} y(t)$ ). Analogously, if  $y$  becomes too small,  $\int h(\lambda) d\mu(\lambda)$  will be the

dominant term. It is immediate from this that  $y$  cannot go to zero or infinity in finite time, and therefore the solution exist globally.

For  $t > t_0$ , we now have  $f = 0$ , and, putting  $\alpha(\lambda) = \gamma(\lambda) - \frac{1}{\lambda}$ ,  $\beta(\lambda) = \delta(\lambda) - \frac{1}{\lambda}$ , we find from (2.3)

$$\int \left\{ \frac{3}{2} \eta \frac{\alpha \dot{\alpha}}{\alpha + \frac{1}{\lambda}} + 3\eta \frac{\beta \dot{\beta}}{\beta + \frac{1}{\lambda}} \right\} d\mu(\lambda) = - \int \left\{ \frac{3}{2} \eta \lambda \frac{\alpha^2}{\alpha + \frac{1}{\lambda}} + 3\eta \lambda \frac{\beta^2}{\beta + \frac{1}{\lambda}} \right\} d\mu(\lambda) \quad (2.8)$$

$$- \left( \int (\alpha(\lambda) - \beta(\lambda)) d\mu(\lambda) \right)^2$$

As we know that  $\gamma(\lambda)$  and  $\delta(\lambda)$  stay positive, the denominators  $\alpha + \frac{1}{\lambda}$  and  $\beta + \frac{1}{\lambda}$  are always positive, and the left side of (2.8) is therefore the derivative of a positive function that decreases along trajectories. (It is easy to prove that  $\alpha$  and  $\beta$  are nice enough for all the integrals to make sense, namely, one sees from (2.2) that  $\lambda g(\lambda)$  and  $\lambda h(\lambda)$  and hence  $\lambda \gamma(\lambda)$  and  $\lambda \delta(\lambda)$  are bounded). As a consequence,  $\alpha$  and  $\beta$  converge to zero exponentially as  $t \rightarrow \infty$  in the  $L^2$ -norm and a fortiori in the  $L^1$ -norm. From the first equation of (2.3) one sees then that  $\dot{y}$  converges to zero exponentially, whence  $y$  must converge to a limit exponentially. Moreover, one easily concludes from the second and third equations of (2.3) that  $\alpha$  and  $\beta$  in fact converge to zero in the  $L^\infty$ -norm and not only in the  $L^2$ -norm. This concludes the proof.

Remark:

It is almost trivial to prove [7] that  $y(\infty) > 1$  if  $f > 0$  and  $y(\infty) < 1$  if  $f < 0$ . Since the equation under study describes the evolution of the length of the filament, if inertia are neglected, this is a result that one would obviously expect. We have no analogue yet for the full problem (0.1), (0.2).

### 3. LOCAL TIME EXISTENCE.

We now turn to the study of (0.1). According to what we have seen in the last chapter, we consider  $u_x(t) = b(t) > 0$  as being given at  $x = \pm 1$ , where  $g$  is a smooth function of  $t$ . We want to reformulate (0.1) in such a way that it fits into the theory of quasilinear parabolic equation. For this purpose we make the following substitutions

$$p = u_x$$

$$q = u_{xx}$$

$$r = \dot{u}$$

$$g_1(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} (u_x(s) - u_x(t)) ds$$

$$g_2(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} \left( \frac{u_{xx}(s)}{u_x^3(s)} - \frac{u_{xx}(t)u_x(s)}{u_x^4(t)} \right) ds$$

$$g_3(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} (u_x^{-2}(s) - u_x^{-2}(t)) ds$$

$$g_4(\lambda) = \int_{-\infty}^t e^{-\lambda(t-s)} \left( u_{xx}(s) - \frac{u_{xx}(t)u_x^2(t)}{u_x^2(s)} \right) ds$$

Equation (0.1) now assumes the following form:

$$\dot{p} = r_x$$

$$\dot{q} = r_{xx}$$

$$\rho \dot{r} = 3\eta p^{-2} r_{xx} - 6\eta p^{-3} q r_x - 2p \int g_2(\lambda) d\mu(\lambda) - \frac{1}{2} \int g_4(\lambda) d\mu(\lambda)$$

$$\dot{g}_1(\lambda) = -\lambda g_1(\lambda) - \frac{r_x}{\lambda} \quad (3.1)$$

$$\dot{g}_2(\lambda) = -\lambda g_2(\lambda) - \frac{r_{xx}}{p^4} \left( g_1(\lambda) + \frac{p}{\lambda} \right) + \frac{4r_x q}{p^5} \left( g_1(\lambda) + \frac{p}{\lambda} \right)$$

$$\dot{g}_3(\lambda) = -\lambda g_3(\lambda) + \frac{2r_x}{\lambda p^3}$$

$$\dot{g}_4(\lambda) = -\lambda g_4(\lambda) - r_{xx} p^2 \left( g_3(\lambda) + \frac{1}{\lambda p^2} \right) - 2r_x q p \left( g_3(\lambda) + \frac{1}{\lambda p^2} \right)$$

with boundary condition  $p = b(t)$ ,  $r_x = \dot{b}(t)$  at  $x = \pm 1$ .

Since  $\dot{p} = r_x$ , the first boundary condition follows from the second, once it is satisfied initially, and we shall ignore it.

We will show that (3.1) can be treated by the Sobolevskii theory. For this we first introduce some notations.  $H^k$  will denote Sobolev spaces of functions on  $[-1,1]$ , and  $L^s(\mu, H^k)$  will denote the space of  $H^k$ -valued functions defined on  $\mathbb{E}$ , which are  $s$ -integrable with respect to the total variation of  $\mu$  in the Bochner sense (for a precise definition, see e.g. [10]). We put  $X_s = H^2 \times (H^1)^2 \times (L^s(\mu, H^1))^4$ . Moreover, in (3.1) we substitute  $\hat{r} = r - \dot{b}(t)x$  and introduce the abbreviation  $y = (p, q, \hat{r}, g_1, g_2, g_3, g_4)$ . We rewrite (3.1) in the form

$$\dot{y} = A(y)y + f(y, t) \quad (3.2)$$

where  $A(y)$  is defined as the following linear operator

$$\begin{aligned} A(y)y' = & (\hat{r}'_x, \hat{r}'_{xx}, \frac{1}{\rho} (3np^{-2}\hat{r}'_{xx} - 6np^{-3}q\hat{r}'_x), -\lambda g'_1(\lambda) - \frac{1}{\lambda}\hat{r}'_x, -\lambda g'_2(\lambda) - (g_1(\lambda) + \frac{p}{\lambda})\frac{\hat{r}_{xx}}{4}, \\ & + \frac{4q}{5}(g_1(\lambda) + \frac{p}{\lambda})\hat{r}'_x, -\lambda g'_3(\lambda) + \frac{2}{\lambda p}\hat{r}'_x, -\lambda g'_4(\lambda) - p^2(g_3(\lambda) + \frac{1}{\lambda p})\hat{r}'_{xx} - 2qp(g_3(\lambda) + \frac{1}{\lambda p})\hat{r}'_x) \end{aligned}$$

with the boundary conditions  $\hat{r}'_x = 0$  at  $x = \pm 1$ .

We shall show that (3.2) satisfies all the requirements of the Sobolevskii theory when regarded as an evolution problem in  $X_s$  ( $1 \leq s < \infty$ ). More precisely, we shall prove

Theorem 3.1:

Let  $1 \leq s < \infty$  be arbitrary. Let  $y_0 = (p_0, q_0, \hat{r}_0, g_{1,0}) \in X_s$  be given such that  $\hat{r}_0 \in H^3$ ,  $\hat{r}_{0,x} = 0$  at  $x = \pm 1$ ,  $\lambda g_{i,0} \in L^s(\mu, H^1)$  and  $\min_{x \in [-1,1]} p_0(x) > 0$ . Then, for some  $T > 0$ , equation (3.2) has a unique solution  $y \in C^1([0, T], X_s)$  such that  $y(0) = y_0$ .

Proof:

We shall deduce the result from Theorem 16.2 in [2] (Theorem 7 in [8] resp.).

For this we have to verify the following conditions stated in [2] as (F1) and (F3)-(F5):

(F1) The operator  $A_0 = A(y_0)$  is densely defined, closed and generates an analytic semigroup.

(F3) For  $v, w$  in a neighborhood of  $y_0$  in  $X_g$  there exists an appropriate  $\beta \in \mathbb{C}$  such that

$$\| (A(v) - A(w)) (A(w) + \beta)^{-1} \| \leq c \|v - w\|$$

with some constant  $c$  independent of  $v$  and  $w$ .

(F4) For  $v, w$  in a neighbourhood of  $y_0$  and  $t, \tau \in [0, T]$  there is some constant  $c$  such that

$$\|f(v, t) - f(w, \tau)\| \leq c(|t - \tau| + \|v - w\|)$$

(F5)  $y_0 \in D(A_0)$

(The conditions in [2] are more general, and we have only formulated the special case applying to our problem).

(F4) and (F5) are trivial consequences of the smoothness of  $b$  and our assumptions on the initial data. (F3) is clear, if it is proved that the  $H^3$ -norm of the  $\hat{r}$ -component of  $(A(w) + \beta)^{-1}y$  can be estimated by  $\|y\|$ . This will be immediate from the arguments leading to (F1) with  $y_0$  replaced by  $w$ .

To prove (F1), consider the equation  $(A_0 + \beta)y = y'$ . In the  $\hat{r}$ -component this leads to

$$\frac{3\eta}{\rho} p_0^{-2} \hat{r}_{xx} - 6\eta p_0^{-3} q_0 \hat{r}_x + \beta \hat{r} = \hat{r}'$$

and the equations for the other components can be trivially resolved once  $\hat{r}$  is known.

It is now a simple consequence of Theorem 19.2 in [2] (which is due to Agmon and Nirenberg [9]) that if  $\beta$  is in a sector not containing the positive real axis, and  $|\beta|$  is large enough, we have an estimate of the form

$$\|\hat{r}\|_{H^3} + |\beta|^{1/2} \|\hat{r}\|_{H^2} + |\beta| \|\hat{r}\|_{H^1} \leq c \|\hat{r}'\|_{H^1}.$$

This concludes the proof.

#### 4. SOLUTIONS FOR SMALL FORCES.

The goal of the present chapter is to establish an analogue of Theorem 2.3 for the equation (0.1), i.e. to prove existence of solutions globally in time for small forces  $f$ .  $f = 0$  now corresponds to the boundary condition  $u_x(t) = b(t) = 1 \Rightarrow \dot{b}(t) = 0$ . In this case (3.1) has the trivial solution  $p = 1, q = 0, r = 0, g_i(\lambda) = 0$ . As a first step we shall study the linearization of (3.1) at this trivial solution with homogeneous boundary conditions  $r_x = 0$ . The linearized equation reads as follows:

$$\begin{aligned}\dot{p} &= r_x \\ \dot{q} &= r_{xx} \\ \dot{r} &= \frac{3\eta}{\rho} r_{xx} - \frac{2}{\rho} \int g_2(\lambda) d\mu(\lambda) - \frac{1}{\rho} \int g_4(\lambda) d\mu(\lambda) \\ \dot{g}_1(\lambda) &= -\lambda g_1(\lambda) - \frac{r_x}{\lambda} \\ \dot{g}_2(\lambda) &= -\lambda g_2(\lambda) - \frac{r_{xx}}{\lambda} \\ \dot{g}_3(\lambda) &= -\lambda g_3(\lambda) + \frac{2r_x}{\lambda} \\ \dot{g}_4(\lambda) &= -\lambda g_4(\lambda) - \frac{r_{xx}}{\lambda}\end{aligned}$$

We abbreviate (4.1) in the form  $\dot{y} = Ay$ . We shall study the spectral properties of  $A$  as an operator in the space  $X_s$  of §3 ( $1 \leq s < \infty$  is again arbitrary). Consider the resolvent equation  $(A - \alpha)y = f = (f_1, f_2, f_3, f_4(\lambda), f_5(\lambda), f_6(\lambda), f_7(\lambda))$ . If  $-\alpha$  is not in the support of  $\mu$ , this equation is immediately resolved with respect to  $p, q$  and  $g_i(\lambda)$ , yielding the following equation for  $r$

$$\frac{3\eta}{\rho} r_{xx} + \frac{3\eta}{\rho} r_{xx} \cdot \int \frac{1}{\lambda(\lambda + \alpha)} d\mu(\lambda) - \alpha r = f_3 - \frac{2}{\rho} \int \frac{f_5(\lambda)}{\lambda + \alpha} d\mu(\lambda) - \frac{1}{\rho} \int \frac{f_7(\lambda)}{\lambda + \alpha} d\mu(\lambda).$$

As noted in §2,  $\int \frac{1}{\lambda(\lambda + \alpha)} d\mu(\lambda)$  has a positive real part for  $\operatorname{Re} \alpha \geq 0$ . Moreover, this expression obviously goes to zero like  $\frac{1}{|\alpha|}$  if  $\alpha \rightarrow \infty$  in any sector  $\{\alpha \in \mathbb{C} \mid -\pi + \varphi + \varepsilon \leq \arg \alpha \leq \pi - \varphi - \varepsilon\}$ ,  $\varphi$  being the angle of assumptions (a) and  $\varepsilon$  any positive number. From these properties it can easily be seen that the following holds:

Proposition 4.1:

$A$  is the generator of an analytic semigroup. Moreover, the spectrum of  $A$  consists of the semi-simple eigenvalue  $0$  and a remainder lying strictly in the left half plane.

Semi-simple here means that the resolvent has a simple pole at  $0$ , or equivalently, that  $R(A) \oplus N(A) = X_S$ ,  $R(A)$  and  $N(A)$  denoting the range and nullspace of  $A$ .

For technical reasons, the spaces  $X_n^\sigma, Y_n^\sigma$  of §2 are not quite appropriate for the study of our present problem, and we shall use the following spaces, which are defined in a very similar manner.

Definition 4.2:

Let  $Z$  be a Banach space. Then  $H^n(\mathbb{R}, Z)$  denotes the spaces of all functions  $\mathbb{R} \rightarrow Z$  whose first  $n$  derivatives are square integrable in the sense of Bochner. Let moreover be

$$\hat{X}_n^\sigma(Z) = \{v \in H^n(\mathbb{R}, Z) \mid e^{\sigma t} v, e^{-\sigma t} v \in H^n(\mathbb{R}, Z)\}$$

$$\hat{Y}_n^\sigma(Z) = \{v : \mathbb{R} \rightarrow Z \mid e^{-\sigma t} v \in H^n(\mathbb{R}, Z), \exists v_\infty \in Z \text{ such that } e^{+\sigma t}(v - v_\infty) \in H^n(\mathbb{R}, Z)\}.$$

Natural norms in  $\hat{X}_n^\sigma$  and  $\hat{Y}_n^\sigma$  are defined in an analogous way as for  $X_n^\sigma, Y_n^\sigma$ . The use of these definitions lies in the following lemma:

Lemma 4.3:

Let the space  $X_2$  and the operator  $A$  be as above, and let  $\sigma > 0$  be small enough. Then the operator

$$y(t) \rightarrow \left(A - \frac{d}{dt}\right)^{-1} y(t)$$

is bounded from  $\hat{X}_n^\sigma(X_2)$  into  $\hat{Y}_n^\sigma(N(A)) \oplus \hat{X}_n^\sigma(R(A) \cap D(A))$ , where  $N(A)$ ,  $R(A)$  and  $D(A)$  denote the nullspace, range and domain of  $A$ , resp.

For the proof, note that since  $X_2$  is a Hilbert space, the norm in  $H^n(\mathbb{R}, X_2)$  can easily be expressed in terms of the Fourier transform, thus reducing the statement of the lemma to estimates on the resolvent of  $A$ . The latter follow from Proposition 4.1. (It is this argument that fails, if  $X_n^\sigma$  is chosen rather than  $\hat{X}_n^\sigma$ ).



With these preliminaries, it is now easy to establish an analogue of Theorem 2.3 for the nonlinear problem (3.1). Again we put  $\tilde{r} = r - \dot{b}(t)x$ , and we put  $y = (p - 1, q, \tilde{r}, g_1, g_2, g_3, g_4)$ . Then (3.1) has the form

$$\dot{y} = Ay + \tilde{f}(y, \dot{b}(t)) \quad (4.2)$$

where  $A$  is the operator studied above.  $\tilde{f}$  is a smooth mapping from

$\hat{Y}_n^\sigma(N(A)) \oplus \hat{X}_n^\sigma(D(A) \cap R(A)) \times \hat{X}_0^n(\mathbb{R})$  into  $\hat{X}_n^\sigma(X_2)$  for any  $n \geq 1$ , and, according to Lemma 4.3,  $(\frac{d}{dt} - A)^{-1}$  is (for  $\sigma$  small enough) a bounded linear mapping from  $\hat{X}_n^\sigma(X_2)$  into  $\hat{Y}_n^\sigma(N(A)) \oplus \hat{X}_n^\sigma(D(A) \cap R(A))$ . The following result is now immediate from the implicit function theorem:

Theorem 4.2:

Let  $\sigma > 0$  be small enough. Then, in a neighbourhood of  $y = 0$ ,  $\dot{b} = 0$  in  $\hat{Y}_n^\sigma(N(A)) \oplus \hat{X}_n^\sigma(D(A) \cap R(A)) \times \hat{X}_n^\sigma(\mathbb{R})$ , equation (4.2) has a unique resolution  $y = y(\dot{b})$ .

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We study the equation $\partial_t^2 u = 3\gamma \frac{\partial^2}{\partial x \partial t} \left( -\frac{1}{u_x} \right) + \frac{\partial}{\partial x} \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds$ <div style="text-align: right;">(continued)</div>		

ABSTRACT (continued)

where  $u(x,t)$  is a real valued function of  $x \in [-1,1]$  and  $t \in \mathbb{R}$ , with the boundary condition

$$3\eta \frac{\partial}{\partial t} \left( -\frac{1}{u_x} \right) + \int_{-\infty}^t a(t-s) \left( \frac{u_x(t)}{u_x^2(s)} - \frac{u_x(s)}{u_x^2(t)} \right) ds = f(t)$$

at  $x = \pm 1$ . This equation is derived as a model for the elongation of thin filaments of polymeric liquids,  $u$  denoting the position of a fluid particle in space,  $a$  the memory kernel, and  $f$  the force acting on the ends of the filament. We study the evolution of  $u$ , assuming the initial condition  $u(x, t = -\infty) = x$ . It is shown that under appropriate conditions on  $a$  and  $f$  the boundary condition can be uniquely resolved with respect to  $u_x$ . The full problem is transformed in such a way that it is approachable by the Sobolevskii theory of quasilinear parabolic equations. This yields the existence of solutions to the initial value problem on sufficiently small time intervals. Moreover, we show that if  $f(t)$  converges to zero exponentially as  $t \rightarrow +\infty$  and is small in an appropriate norm, there exists a solution globally in time, which approaches a stationary limit as  $t \rightarrow +\infty$ .